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so that $$w = e^{-\frac{1}{2}\int P dx}\, u,$$
is the transformation which converts

$$\frac{d^2w}{dx^2} + P\frac{dw}{dx} + Q = 0 \qquad (1)$$

into the form (14) which lacks the second term, and Q_0 has the value given in equation (13).

◆

SOLUTIONS OF EXERCISES.

60 and 61

Divide a sphere by a secant plane into two parts whose surfaces (volumes) shall be as 1 : 3.

SOLUTION.

Let the secant plane divide the diameter normal to it in the ratio $m : n$. The heights and surfaces of the two zones are respectively

$$\frac{n}{m+n}D, \quad \frac{m}{m+n}D; \quad \text{and} \quad \frac{\pi n}{m+n}D^2, \quad \frac{\pi m}{m+n}D^2.$$

The area of the common base of the two segments is

$$\frac{\pi mn}{(m+n)^2}D^2.$$

Which values under the given condition, give

1. For the surfaces,

$$3\left[\frac{\pi n D^2}{m+n} + \frac{\pi mn D^2}{(m+n)^2}\right] = \frac{\pi m D^2}{m+n} + \frac{\pi mn D^2}{(m+n)^2};$$

from which $$\left(\frac{m}{n}\right)^2 - 4\left(\frac{m}{n}\right) - 3 = 0,$$

$$\frac{m}{n} = \sqrt{7} + 2 = 4.64575+.$$

2 For the volumes, the mid-areas being

$$\frac{\pi n D^2}{4}\frac{2m+n}{(m+n)^2} \quad\text{and}\quad \frac{\pi m D^2}{4}\frac{2n+m}{(m+n)^2},$$

the mean areas are

$$\frac{\pi n D^2}{6(m+n)^2}(3m+n) \quad\text{and}\quad \frac{\pi m D^2}{6(m+n)^2}(3n+m);$$

whence
$$3\,\frac{\pi n^2 D^2(3m+n)}{6(m+n)^2} = \frac{\pi m^2 D^2(3n+m)}{6(m+n)^2},$$

from which
$$\left(\frac{m}{n}\right)^3 + 3\left(\frac{m}{n}\right)^2 - 9\frac{m}{n} - 3 = 0,$$

$$\frac{m}{n} = 2.06416+.$$

The condition $n < m < (m+n)$ selects the roots. [*W. H. Echols.*]

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Each of four equal circles touches two of the others. Find the average area included between them. [*Artemas Martin.*]

SOLUTION.

Let A, B, C, D be the centres of the four equal circles, r their radius, and let $\varphi = \angle BAC = \angle BDC$; then $AB = AC = BD = CD = 2r$, and area of triangle $BAC = 2r^2 \sin\varphi$ = area of triangle BDC. Since the sum of the angles $A, B, C, D = 4\pi$, the sum of the areas of the four sectors $= \pi r^2$. Hence the area enclosed by the circles is $4r^2\sin\varphi - \pi r^2$. The limits of φ are $\frac{1}{2}\pi$ and $\frac{1}{3}\pi$. Let Δ = the average area required, and we have

$$\Delta = \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}(4r^2\sin\varphi - \pi r^2)\,d\varphi \div \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} d\varphi$$

$$= \frac{6r^2}{\pi}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}(4\sin\varphi - \pi)\,d\varphi$$

$$= \frac{6r^2}{\pi}\Big[-4\cos\varphi - \pi\varphi\Big]_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}$$

$$= \frac{r^2}{\pi}(12 - \pi^2).$$

[*Artemas Martin; C. A. Waldo; Henry Heaton; E. L. Stabler.*]

111

A CIRCLE is inscribed in an isosceles triangle; another in the space between the first circle and the vertex, and so on *ad infinitum*. What is the vertical angle when the sum of the areas of the circles is $1/n$ of the area of the triangle?

[*L. G. Weld.*]

SOLUTION.

Let $2a$ be the base, $2x$ the vertical angle, Δ the area of the triangle; ρ the radius, and C the area of the first inscribed circle.

Then
$$\Delta = a^2 \cot x,$$

$$\rho = a \tan (45^\circ - \tfrac{1}{2}x) = a \frac{1 - \sin x}{\cos x},$$

$$C = \pi a^2 \tan^2 (45^\circ - \tfrac{1}{2}x) = \pi a^2 \frac{1 - \sin x}{1 + \sin x}.$$

The second circle is inscribed in a triangle having the same vertical angle $2x$, and a base $2a' = 2a \dfrac{a \cot x - 2\rho}{a \cot x} = 2a \dfrac{1 - \sin x}{1 + \sin x}$. Its area C' is therefore

$$C' = \pi a'^2 \frac{1 - \sin x}{1 + \sin x} = \pi a^2 \left(\frac{1 - \sin x}{1 + \sin x}\right)^3 = C \left(\frac{1 - \sin x}{1 + \sin x}\right)^2.$$

The same reasoning applies to the following circles.

The condition of the problem,

$$\frac{1}{n} \Delta = C + C' + C'' + \dots,$$

thus becomes

$$\frac{1}{n} a^2 \cot x = \pi a^2 \frac{1 - \sin x}{1 + \sin x} \left[1 + \left(\frac{1 - \sin x}{1 + \sin x}\right)^2 + \left(\frac{1 - \sin x}{1 + \sin x}\right)^4 + \dots\right]$$

$$= \pi a^2 \frac{\cos^2 x}{4 \sin x};$$

hence
$$\cos x = \frac{4}{n\pi}, \quad \text{or } 2x = 2 \cos^{-1} \frac{4}{n\pi}.$$

[*Alexander Ziwet; L. G. Weld; E. L. Stabler.*]

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GENERALIZATION II.

To inscribe in a given triangle ABC a triangle PQR whose sides QR, RP, PQ pass respectively through three given points O_p, O_q, O_r.

SOLUTION.

To solve this more general problem, draw through O_p any three straight lines cutting AB in P_1, P_2, P_3 and BC in Q_1, Q_2, Q_3; draw O_qQ_1, O_qQ_2, O_qQ_3, which will meet CA in three points R_1, R_2, R_3; finally, join O_r to R_1, R_2, R_3, and let these joining lines meet AB in P_1', P_2', P_3'. The pencils O_p and O_q, as well as O_q and O_r, being in perspective position, the ranges P, Q, R, P' are projective. The two collinear projective ranges P and P', on AB, will, in general, have two self-corresponding points, which are the solutions of the problem. The construction of the self-corresponding points of two such ranges, is a well-known fundamental problem of Modern Geometry and can always be effected by means of an auxiliary circle.*

Now, let O_p, O_q, O_r be the points at infinity in the directions perpendicular to AB, BC, CA respectively, and the general problem reduces to the particular case proposed by Mr. Lane. In this case, the ranges P, Q, R, P' are *similar*, and, therefore, need only two points each for their determination. We may then proceed as follows: Selecting as the arbitrary points P_1, P_2 on AB, the point B, and the foot of the perpendicular from C on AB, we have at once for Q_1, Q_2 the points B and C, since they are the intersections with BC of the perpendiculars to AB erected in P_1, P_2. The perpendicular to BC in B intersects CA in R_{D} while C is the point R_2. Lastly, the perpendiculars to CA erected in R_1 and C meet AB in two points P_1', P_2', which form a range similar to, and collinear with, B, P_2. The self-corresponding point (or centre of similitude) can, of course, be constructed linearly, without the aid of a circle, for instance, as follows: Draw a parallel to AB, and from P_1, P_2 drop perpendiculars on it, meeting it in P_1'', P_2''; join $P_1'P_1''$, $P_2'P_2''$, and from the intersection of these lines drop a perpendicular on AB; it will meet this line in the self-corresponding point of the ranges.

[*Alexander Ziwet.*]

116

FIND the value of $\int Qdx$ where

$$Q = \cos(a_1x + b_1)\cos(a_2x + b_2)\ .\ .\ .\ \cos(a_nx + b_n),$$

$a_1, a_2, \ .\ .\ .\ a_n$, and $b_1, b_2, \ .\ .\ .\ b_n$ being constants. [*A. Hall.*]

SOLUTION.

Applying successively the formulæ,

$$\cos A \cos B = \tfrac{1}{2}\left[\cos(A + B) + \cos(A - B)\right],$$

$$\cos(A \pm B)\cos C = \tfrac{1}{2}\left[\cos(A \pm B + C) + \cos(A \pm B - C)\right],$$

etc., etc.,

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*See, for instance, L. Cremona's *Projective Geometry*, translated by Leudesdorf, Oxford 1885, p. 169.

we obtain

$$Q = (\tfrac{1}{2})^n\, \Sigma \cos\left[(a_1 \pm a_2 \;.\;.\;.\; \pm a_n)\,x + b_1 \pm b_2 \;.\;.\;.\; \pm b_n\right],$$

where the signs $\pm$ are independent of one another and include all possible variations; provided, each b has the same sign as the corresponding a; whence immediately,

$$\int Q dx = \frac{1}{2^n\,(a_1 \pm a_2 \;.\;.\;.\; \pm a_n)}\, \Sigma \sin\left[(a_1 \pm a_2 \ldots \pm a_n)\,x + b_1 \pm b_2 \ldots \pm b_n\right].$$

In case, however, any coefficient of x in Q vanishes, we have for the corresponding term in $\int Q dx$

$$(\tfrac{1}{2})^n \int \cos\,(b_1 \pm b_2 \;.\;.\;.\; \pm b_n)\,dx = (\tfrac{1}{2})^n\, x \cos\,(b_1 \pm b_2 \;.\;.\;.\; \pm b_n).$$

[*Ormond Stone.*]

122

IF $P_1P_2P_3$ be a triangle inscribed in an ellipse, the co-ordinates of the point of concourse of its altitudes are given by the relations,

$$2ax = (a^2 + b^2)\,(\cos p_1 + \cos p_2 + \cos p_3) - (a^2 - b^2)\cos\,(p_1 + p_2 + p_3),$$

$$2by = (a^2 + b^2)\,(\sin p_1 + \sin p_2 + \sin p_3) - (a^2 - b^2)\sin\,(p_1 + p_2 + p_3);$$

where p_1, p_2, p_3 are the eccentric anomalies of the vertices. Prove these relations, and find the values of x, y when for a given P the area of $P_1P_2P_3$ is greatest. Show that the locus of the orthocentre of all such triangles is also an ellipse.

[*W. M. Thornton.*]

SOLUTION.

1. The abscissa of the circumcentre is, by Exercise 98,

$$x_1 = \frac{c^2}{a}\cos\tfrac{1}{2}\,(p_2 + p_3)\cos\tfrac{1}{2}\,(p_3 + p_1)\cos\tfrac{1}{2}\,(p_1 + p_2)$$

$$= \frac{c^2}{4a}\left[\cos p_1 + \cos p_2 + \cos p_3 + \cos\,(p_1 + p_2 + p_3)\right],$$

by easy reductions. The abscissa of the centroid is

$$x_2 = \tfrac{1}{3}a\,(\cos p_1 + \cos p_2 + \cos p_3).$$

The orthocentre divides the line joining these points in the ratio 3 : 2 externally; that is,

$$x = 3x_2 - 2x_1$$

$$= a\,(\cos p + \cos p_2 + \cos p_3)$$

$$- \frac{c^2}{2a}\left[\cos p_1 + \cos p_2 + \cos p_3 + \cos\,(p_1 + p_2 + p_3)\right],$$

or $2ax = (a^2 + b^2)\,(\cos p_1 + \cos p_2 + \cos p_3) - (a^2 - b^2)\cos\,(p_1 + p_2 + p_3).$

The ordinate can be found in the same way.

2. The area of the triangle (Ex. 98) is

$$\varDelta = 2ab \sin \tfrac{1}{2} (p_2 - p_3) \sin \tfrac{1}{2} (p_3 - p_1) \sin \tfrac{1}{2} (p_1 - p_2),$$

the differentiation of which, since p_1, p_2, p_3 are independent of one another, gives, after reduction,

$$\tan \tfrac{1}{2} (p_1 - p_2) = \tan \tfrac{1}{2} (p_2 - p_3) = \tan \tfrac{1}{2} (p_3 - p_1).$$

Whence for the maximum

$$p_1 - p_2 = p_2 - p_3 = p_3 - p_1 = \tfrac{2}{3} n\pi,$$

where n is an arbitrary integer. Consequently

$$\cos p_1 + \cos p_2 + \cos p_3 = \cos p_1 + \cos (p_1 + \tfrac{2}{3} n\pi) + \cos (p_1 + \tfrac{4}{3} n\pi) = 0,$$

$$\sin p_1 + \sin p_2 + \sin p_3 = \sin p_1 + \sin (p_1 + \tfrac{3}{3} n\pi) + \sin (p_1 + \tfrac{4}{3} n\pi) = 0,$$

$$p_1 + p_2 + p_3 = 3p_1 + 2n\pi;$$

and the co-ordinates of the orthocentre are given by the relations,

$$2ax = -(a^2 - b^2) \cos 3p_1,$$

$$2by = -(a^2 - b^2) \sin 3p_1.$$

3. Whence $a^2x^2 + b^2y^2 = \frac{1}{4} (a^2 - b^2)^2$, the equation to an ellipse.

[*W. M. Thornton.*]

123

FIND an expression for the area of a quadrilateral inscribed in an ellipse in terms of the eccentric anomalies of its vertices and the axes of the curve.

[*R. H. Graves.*]

SOLUTION I.

Let the equation to the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$; let the co-ordinates of the vertices taken in order be (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) and let the eccentric angles of the same points be α, β, γ, δ.

The area of the quadrilateral

$$= \pm \tfrac{1}{2} (y_1x_2 - y_2x_1 + y_2x_3 - y_3x_2 + y_3x_4 - y_4x_3 + y_4x_1 - y_1x_4)$$

$$= \pm \tfrac{1}{2} \begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ y_1 & y_2 & y_3 & y_4 \\ x_1 & x_2 & x_3 & x_4 \end{vmatrix}$$

$$= \pm \tfrac{1}{2} ab \begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \sin\alpha & \sin\beta & \sin\gamma & \sin\delta \\ \cos\alpha & \cos\beta & \cos\gamma & \cos\delta \end{vmatrix}$$

$$= \pm \tfrac{1}{2} ab \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \sin\alpha - \sin\gamma & \sin\beta - \sin\delta & \sin\gamma & \sin\delta \\ \cos\alpha - \cos\gamma & \cos\beta - \cos\delta & \cos\gamma & \cos\delta \end{vmatrix}$$

$$= \pm \tfrac{1}{2} ab \begin{vmatrix} 2\cos\tfrac{1}{2}(\alpha+\gamma)\sin\tfrac{1}{2}(\alpha-\gamma) & 2\cos\tfrac{1}{2}(\beta+\delta)\sin\tfrac{1}{2}(\beta-\delta) \\ 2\sin\tfrac{1}{2}(\alpha+\gamma)\sin\tfrac{1}{2}(\gamma-\alpha) & 2\sin\tfrac{1}{2}(\beta+\delta)\sin\tfrac{1}{2}(\delta-\beta) \end{vmatrix}$$

$$= \pm\, 2ab \sin\tfrac{1}{2}(\alpha-\gamma)\sin\tfrac{1}{2}(\beta-\delta) \begin{vmatrix} \cos\tfrac{1}{2}(\alpha+\gamma) & \cos\tfrac{1}{2}(\beta+\delta) \\ \sin\tfrac{1}{2}(\alpha+\gamma) & \sin\tfrac{1}{2}(\beta+\delta) \end{vmatrix}$$

$$= \pm\, 2ab \sin\tfrac{1}{2}(\alpha-\gamma)\sin\tfrac{1}{2}(\beta-\delta)\sin\tfrac{1}{2}(\alpha+\gamma-\beta-\delta).$$

COR. 1. If $\gamma = \delta$, the expression becomes $\pm\, 2ab \sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha)$, which agrees with the formula for the area of an inscribed triangle given in Salmon's *Conic Sections*, Art. 231, Ex. 5.

COR. 2. If the vertices lie also on the circumference of a circle, $\alpha+\beta+\gamma+\delta = 2n\pi$.* Then the area is $\pm\, 2ab \sin\tfrac{1}{2}(\alpha-\gamma)\sin\tfrac{1}{2}(\beta-\delta)\sin(\beta+\delta)$.

COR. 3. If the normals at the four vertices meet in a point, $\alpha+\beta+\gamma+\delta = (2n+1)\pi$.† Then the area is $\pm\, 2ab \sin\tfrac{1}{2}(\alpha-\gamma)\sin\tfrac{1}{2}(\beta-\delta)\cos(\beta+\delta)$.

[*R. H. Graves.*]

SOLUTION II.

From Exercise 98

$$P_1P_3P_2 = \quad 2ab \sin\tfrac{1}{2}(p_1-p_3)\sin\tfrac{1}{2}(p_3-p_2)\sin\tfrac{1}{2}(p_2-p_1),$$

$$P_1P_3P_4 = -\,2ab \sin\tfrac{1}{2}(p_1-p_3)\sin\tfrac{1}{2}(p_3-p_4)\sin\tfrac{1}{2}(p_4-p_1).$$

But

$$2\sin\tfrac{1}{2}(p_3-p_2)\sin\tfrac{1}{2}(p_2-p_1) = \cos\tfrac{1}{2}(p_1-2p_2+p_3) + \cos\tfrac{1}{2}(p_3-p_1),$$

$$2\sin\tfrac{1}{2}(p_3-p_4)\sin\tfrac{1}{2}(p_4-p_1) = \cos\tfrac{1}{2}(p_1-2p_4+p_3) + \cos\tfrac{1}{2}(p_3-p_1),$$

and

$$\cos\tfrac{1}{2}(p_1-2p_2+p_3) - \cos\tfrac{1}{2}(p_1-2p_4+p_3) = 2\sin\tfrac{1}{2}(p_1-p_2+p_3-p_4)\sin\tfrac{1}{2}(p_2-p_4).$$

Therefore

$$P_1P_2P_3P_4 = 2ab \sin\tfrac{1}{2}(p_1-p_3)\sin\tfrac{1}{2}(p_2-p_4)\sin\tfrac{1}{2}(p_1-p_2+p_3-p_4).$$

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*See Smith's *Conic Sections*, Art. 186, Ex. 1.

†Ibid., Art. 198.

The negative sign is given to the value of $P_1P_3P_4$ because the perimeter is described in the sense opposite to that pursued in describing $P_1P_3P_2$.

[*W. M. Thornton.*]

SOLUTION III.

Adopting the notation of Exercise 98 the area of the quadrilateral can be found by adding the triangle $P_1P_2P_3$ to $P_1P_3P_4$; or by the sum of the triangles $P_1P_2P_4$ and $P_2P_3P_4$. Hence we have the two values of the area:

$$A = 2ab \sin\tfrac{1}{2}(p_3 - p_1)\left[\sin\tfrac{1}{2}(p_4 - p_1)\sin\tfrac{1}{2}(p_4 - p_3) + \sin\tfrac{1}{2}(p_2 - p_1)\sin\tfrac{1}{2}(p_3 - p_2)\right],$$

$$A = 2ab \sin\tfrac{1}{2}(p_4 - p_2)\left[\sin\tfrac{1}{2}(p_4 - p_3)\sin\tfrac{1}{2}(p_3 - p_2) + \sin\tfrac{1}{2}(p_2 - p_1)\sin\tfrac{1}{2}(p_4 - p_1)\right].$$

We may write therefore

$$A = 2ab \sin\tfrac{1}{2}(p_3 - p_1)\sin\tfrac{1}{2}(p_4 - p_2) \times \frac{\sin\frac{1}{2}(p_4 - p_1)\sin\frac{1}{2}(p_4 - p_3) + \sin\frac{1}{2}(p_2 - p_1)\sin\frac{1}{2}(p_3 - p_2)}{\sin\frac{1}{2}(p_4 - p_2)}.$$

Now

$$\begin{aligned}\sin\tfrac{1}{2}(p_4 - p_1) &= \sin\tfrac{1}{2}(p_4 - p_2 + p_2 - p_1)\\ &= \sin\tfrac{1}{2}(p_4 - p_2)\cos\tfrac{1}{2}(p_2 - p_1) + \cos\tfrac{1}{2}(p_4 - p_2)\sin\tfrac{1}{2}(p_2 - p_1),\\ \sin\tfrac{1}{2}(p_4 - p_3) &= \sin\tfrac{1}{2}(p_4 - p_2 + p_2 - p_3)\\ &= \sin\tfrac{1}{2}(p_4 - p_2)\cos\tfrac{1}{2}(p_2 - p_3) + \cos\tfrac{1}{2}(p_4 - p_2)\sin\tfrac{1}{2}(p_2 - p_3);\end{aligned}$$

whence $\sin\frac{1}{2}(p_4 - p_1)\sin\frac{1}{2}(p_4 - p_3)$

$$\begin{aligned}&= \sin\tfrac{1}{2}(p_4 - p_2)^2\cos\tfrac{1}{2}(p_2 - p_3)\cos\tfrac{1}{2}(p_2 - p_1)\\ &+ \sin\tfrac{1}{2}(p_4 - p_2)\cos\tfrac{1}{2}(p_4 - p_2)\cos\tfrac{1}{2}(p_2 - p_3)\sin\tfrac{1}{2}(p_2 - p_1)\\ &+ \sin\tfrac{1}{2}(p_4 - p_2)\cos\tfrac{1}{2}(p_4 - p_2)\sin\tfrac{1}{2}(p_2 - p_3)\cos\tfrac{1}{2}(p_2 - p_1)\\ &+ \cos\tfrac{1}{2}(p_4 - p_2)^2\sin\tfrac{1}{2}(p_2 - p_3)\sin\tfrac{1}{2}(p_2 - p_1).\end{aligned}$$

The second and third of these terms give

$$\sin\tfrac{1}{2}(p_4 - p_2)\cos\tfrac{1}{2}(p_4 - p_2)\sin\tfrac{1}{2}(p_2 - p_3 + p_2 - p_1).$$

Changing $\cos^2$ to $1 - \sin^2$ the last term in A is destroyed, and we have from the first and fourth terms of the product,

$$\sin\tfrac{1}{2}(p_4 - p_2)^2\cos\tfrac{1}{2}(p_2 - p_3 + p_2 - p_1).$$

Dividing out $\sin\frac{1}{2}(p_4 - p_2)$, the last factor of A becomes

$$\sin\tfrac{1}{2}(p_4 - p_2)\cos\tfrac{1}{2}(p_2 - p_3 + p_2 - p_1) + \cos\tfrac{1}{2}(p_4 - p_2)\sin\tfrac{1}{2}(p_2 - p_3 + p_2 - p_1),$$

or $$\sin\tfrac{1}{2}(p_4 - p_3 + p_2 - p_1).$$

Hence the area of the quadrilateral is

$$A = 2ab \sin \tfrac{1}{2}(p_3 - p_1) \sin \tfrac{1}{2}(p_4 - p_2) \sin \tfrac{1}{2}(p_4 - p_3 + p_2 - p_1).$$

[*A. Hall.*]

[Prof. Hall furnished the result contained in Exercise 123 independently of Prof. Graves, but later.]

124

THE area of the curve

$$r^2 = \frac{(a^2 - b^2)^2 \sin^2\theta \cos^2\theta}{a^2 \sin^2\theta + b^2 \cos^2\theta}$$

is one-half that of the circumscribing circle. [*R. H. Graves.*]

SOLUTION I.

It may be shown, from Art. 173 of Williamson's *Differential Calculus*, that the curve is the pedal of the evolute of the ellipse, the centre being the pedal origin; hence its area is $\frac{1}{2}\pi(a - b)^2$.* From the nature of its generation and from the result of Exercise 93 it appears that the radius of its circumscribing circle is $a - b$. Hence the truth of the proposition. [*R. H. Graves.*]

SOLUTION II.

$$\begin{aligned}
2d\Omega &= \frac{c^4 \sin^2\theta \cos^2\theta}{a^2 \sin^2\theta + b^2 \cos^2\theta}\, d\theta \\
&= \frac{c^4 \sin^2\theta \cos^2\theta}{c^2 \sin^2\theta + b^2}\, d\theta \\
&= c^2 \cos^2\theta\, d\theta + b^2 d\theta \frac{a^2b^2}{b^2 + c^2 \sin^2\theta}\, d\theta \\
&= c^2 \cos^2\theta\, d\theta + b^2 d\theta - \frac{a^2b^2}{a^2 \sin^2\theta + b^2 \cos^2\theta}\, d\theta \\
&= \tfrac{1}{2}(a^2 + b^2)\, d\theta + \tfrac{1}{2} c^2 \cos 2\theta - \frac{a^2b^2}{a^2 \tan^2\theta + b^2}\, d \tan\theta;
\end{aligned}$$

$$\therefore\ 2\Omega = \tfrac{1}{2}(a^2 + b^2)\theta + \tfrac{1}{4}c^2 \sin 2\theta - ab \tan^{-1}\left(\frac{a}{b}\tan\theta\right).$$

The limits for one quadrant are 0 and $\frac{1}{2}\pi$.

Hence

$$\begin{aligned}
\tfrac{1}{2}\Omega &= \tfrac{1}{4}\pi(a^2 + b^2) - \tfrac{1}{2}\pi\, ab \\
&= \tfrac{1}{4}\pi(a - b)^2; \\
\therefore\ \Omega &= \tfrac{1}{2}\pi(a - b)^2,
\end{aligned}$$

which was to be proved. [*Wm. M. Thornton.*]

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* See Williamson's *Integral Calculus*, Art. 142.

126

FROM the centre of each of two equal coins a coin is cut at random. If one of these random coins be placed on the other at random in a horizontal position, what is the probability that the top coin will not fall off, supposing one coin just as likely to be placed on top as the other? [*Artemas Martin.*]

SOLUTION.

Let r be the radius of the given equal coins, and x and y the respective radii of the random coins cut from them.

If the coin whose radius is x be placed on top, the chance that it will not fall off is $\frac{\pi x^2}{\pi (x+y)^2} = \frac{x^2}{(x+y)^2}$. If the coin whose radius is y be placed on top, the chance that it will not fall off is $\frac{y^2}{(x+y)^2}$. The total chance, therefore, that the top coin will not fall off for a particular, or given, value of each x and y is $\frac{x^2+y^2}{2(x+y)^2}$.

The probability required is the sum of all the separate probabilities resulting from all possible values of x and y.

$$\therefore\ p = \int_0^r\int_0^r \frac{(x^2+y^2)\,dxdy}{2(x+y)^2} \div \int_0^r\int_0^r dxdy$$

$$= \frac{1}{2r^2}\int_0^r\int_0^r \frac{(x^2+y^2)\,dxdy}{(x+y)^2}$$

$$= \frac{1}{2r^2}\int_0^r \left[(2x+r) - \frac{2x^2}{x+r} + 2x\log x - 2x\log(x+r)\right]dx$$

$$= \frac{1}{2r^2}\Big[2rx + x^2\log x - (x^2+r^2)\log(x+r)\Big]_0^r$$

$$= 1 - \log 2.$$

[*Artemas Martin.*]

129

THE two equal sides of an isosceles triangle move on two fixed points. Required the equation of the locus of the centre of gravity of the triangle. [*Yale Problems.*]

SOLUTION.

The locus of the vertex of the triangle is that segment of the circle, on the base determined by the two fixed points, which contains the vertical angle of the triangle.

Its centroid lies on the bisectrix of this angle at a constant distance from the vertex.

The bisectrix passes constantly through a fixed point on the circumference of the circle.

Hence the locus is a circular conchoid.

Referred to the fixed point as pole, and the perpendicular bisector of the base of the segment as axis, its polar equation is of the form

$$\rho = 2R \cos \theta - c.$$

[*Wm. Symmonds.*]

132

A BROKEN line, *ABCDE* . . . is drawn in a plane, having its angles equal and the concavity always on the same side. Each of the successive parts *BC*, *CD*, *DE*, etc., is half as long as the preceding. The length and direction of *AB* are given and the common angle. Required the direction and distance from *A* of the point to which the end of the line approaches, when the construction as described is continued indefinitely. [*Yale Problems.*]

SOLUTION.

Using A as origin, and AB as axis we find the co-ordinates of the terminal point to be

$$x = a + \tfrac{1}{2}a \cos \alpha + \tfrac{1}{4}a \cos 2\alpha + \tfrac{1}{8}a \cos 3\alpha + \ldots,$$

$$y = \tfrac{1}{2}a \sin \alpha + \tfrac{1}{4}a \sin 2\alpha + \tfrac{1}{8}a \sin 3\alpha + \ldots;$$

whence

$$x + iy = a + \tfrac{1}{2}ae^{i\alpha} + \tfrac{1}{4}ae^{2i\alpha} + \tfrac{1}{8}ae^{3i\alpha} + \ldots$$

$$= \frac{a}{1 - \frac{1}{2}e^{i\alpha}}$$

$$= \frac{2a}{(2 - \cos \alpha) i \sin \alpha}$$

$$= 2a \frac{(2 - \cos \alpha) + i \sin \alpha}{(2 - \cos \alpha)^2 + \sin^2 \alpha}.$$

Therefore

$$x = 2a \frac{2 - \cos \alpha}{5 - 2 \cos \alpha}, \quad y = 2a \frac{\sin \alpha}{5 - 2 \cos \alpha},$$

and the required distance and direction are

$$\rho = \sqrt{(x^2 + y^2)} = \frac{2a}{\sqrt{(5 - 2 \cos \alpha)}},$$

$$\theta = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{\sin \alpha}{2 - \cos \alpha}$$

[*L. G. Weld.*]

134

LET ρ_a, ρ_b, ρ_c, ρ_d, ρ_e be five vectors drawn from a common but undetermined origin to five given points A, B, C, D, E. Find five scalars a, b, c, d, e such that the vector equation

$$a\rho_a + b\rho_b + c\rho_c + d\rho_d + e\rho_e = 0$$

shall hold true for all positions of the origin. [*Yale Problems.*]

SOLUTION.

Projection on the axes gives

$$\begin{aligned} a(x-x_a)+b(x-x_b)+c(x-x_c)+d(x-x_d)+e(x-x_e)&=0,\\ a(y-y_a)+b(y-y_b)+c(y-y_c)+d(y-y_d)+e(y-y_e)&=0,\\ a(z-z_a)+b(z-z_b)+c(z-z_c)+d(z-z_d)+e(z-z_e)&=0; \end{aligned}$$

relations which must hold for all values of x, y, z.

Therefore
$$\begin{aligned} ax_a + bx_b + cx_c + dx_d + ex_e &= 0,\\ ay_a + by_b + cy_c + dy_d + ey_e &= 0,\\ az_a + bz_b + cz_c + dz_d + ez_e &= 0,\\ a + b + c + d + e &= 0. \end{aligned}$$

And a, b, c, d, e must be proportional to the minors of

$$\left\| \begin{matrix} x_a & x_b & x_c & x_d & x_e \\ y_a & y_b & y_c & y_d & y_e \\ z_a & z_b & z_c & z_d & z_e \\ 1 & 1 & 1 & 1 & 1 \end{matrix} \right\|$$

[*W. M. Thornton.*]

135

A UNIFORM elastic cord weighs 100 ounces. Its length when unstretched is 100 feet. If it is laid on a smooth table and stretched, its length increases one foot for every ounce of force applied. It is thrown over two pullies twenty feet apart, and on the same level, so that the central loop balances the pendent ends. Discuss the curve and the tensions. [*Yale Problems.*]

SOLUTION.

Let H denote the	uniform horizontal tension in the arc of the catenary,
w	weight of the arc measured from the lowest point to (x, y),
t	tension in the arc at (x, y),
s	length of the arc whose weight is w.

The equations of equilibrium are

$$\frac{dx}{H}=\frac{dy}{w}=\frac{ds}{t},\quad t^2=H^2+w^2.$$

And since the original length of the arc ds is by hypothesis $ds/(1+\frac{t}{100})$,

$$ds=dw\,(1+\tfrac{t}{100}).$$

Hence, we find for the differential equations of the arc

$$dx=\frac{H}{100}dw+H\frac{dw}{\sqrt{(H^2+w^2)}},$$

$$dy=\frac{1}{100}wdw+\frac{wdw}{\sqrt{(H^2+w^2)}}.$$

The integral form of these equations is with the origin at the lowest point of the arc

$$x=\frac{Hw}{100}+H\log\frac{w+\sqrt{(H^2+w^2)}}{H},$$

$$y=\frac{w^2}{200}+\sqrt{(H^2+w^2)}-H.$$

These define the form of the arc.

To determine H we observe that at the pulley the values of t and w are connected by the relation

$$T+W=50;$$

$$\therefore\quad W=25-\frac{H^2}{100},\quad T=25+\frac{H^2}{100},$$

and the abscissa of the pulley is by hypothesis 10.

Hence we have $\qquad \log H=\log 50+0.25-\dfrac{10}{H}-\dfrac{H^2}{10000}.$

This transcendental equation gives approximately

$$H=3.407;$$

$$\therefore\quad W=24.884,\quad T=25\,116.$$

To find the stretched length of the pendent cord, we have from considerations analogous to those used above

$$dP=dT+\tfrac{1}{100}\,TdT,$$

$$P=T+\tfrac{1}{200}\,T^2$$

$$=28.270$$

For the length of the half of the stretched arc we have

$$ds = dw + \tfrac{1}{100}\sqrt{(H^2 + w^2)}\,dw,$$

$$S = W + \frac{1}{200} W\sqrt{(H^2 + W^2)} + \frac{H^2}{200}\log\frac{W + \sqrt{(H^2 + W^2)}}{H}$$

$$= W + \frac{1}{200} TW + \frac{1}{200} H^2 \log\frac{50}{H}$$

$$= 28.164.$$

[*W. M. Thornton.*]

EXERCISES.

SELECTED FROM CASEY'S CONIC SECTIONS.

145

FIND the equation to the circle through the feet of the normals from (h, k) to the parabola $y^2 = 2px$.

146

FIND the condition that the chords AB, CD in an ellipse should meet the transverse axis in points equidistant from the centre, the points A, B, C, D being given by their eccentric anomalies α, β, γ, δ.

147

THE point P on an ellipse is the projection of Q on the circle on the transverse axis. P', Q' are the points diametrically opposite. Find the area of the parallelogram formed by the four points in terms of the eccentric anomaly of P.

148

FIND the area of the parallelogram formed by tangents at the same points.

149

FROM a point P whose distances from the foci of an ellipse are r, r' tangents are drawn to the curve. Find the angle between them.

150

SHOW that the equation to the pair of tangents from $x'y'$ to $b^2x^2 + a^2y^2 - a^2b^2$ may be written

$$a^2(x - x')^2 + b^2(y - y')^2 = (xy' - x'y)^2.$$